REPRESENTATIONS OF RESIDUALLY FINITE GROUPS BY ISOMETRIES OF THE URYSOHN SPACE

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ABSTRACT. As a consequence of Kirchberg's work, Connes' Embedding Conjecture is equivalent to the property that every homomorphism of the group $F_\infty \times F_\infty$ into the unitary group $U(\ell^2)$ with the strong topology is pointwise approximated by homomorphisms with a precompact range. In this form, the property (which we call Kirchberg's property) makes sense for an arbitrary topological group. We establish the validity of the Kirchberg property for the isometry group $\mathrm{Iso}\,(\mathbb{U})$ of the universal Urysohn metric space $\mathbb{U}$ as a consequence of a stronger result: every representation of a residually finite group by isometries of $\mathbb{U}$ can be pointwise approximated by representations with a finite range. This brings up the natural question of which other concrete infinite-dimensional groups satisfy the Kirchberg property.

1. INTRODUCTION

Our motivation comes from theory of operator algebras. Recall that a C^*-algebra is *residually finite dimensional* if it admits a separating family of $*$-homomorphisms into finite-dimensional matrix algebras. The question of describing those countable discrete groups Γ for which the full group C^*-algebra, $C^*(\Gamma)$, is residually finite-dimensional is of considerable interest. For instance, Kirchberg's work [13] implies that the validity of Connes Embedding Conjecture is equivalent to the residual finite dimensionality of the algebra $C^*(F_2 \times F_2)$.

A group Γ is *residually finite* if it admits a separating family of homomorphisms to finite groups, or, equivalently, if the intersection of all subgroups of finite index is reduced to the unity. In view of Malcev's theorem stating that finitely generated subgroups of linear groups are residually finite [14], for a finitely generated group Γ residual finite-dimensionality of $C^*(\Gamma)$ implies residual finiteness of Γ. However, as shown by Bekka [1], this necessary condition on Γ is not sufficient.

It is easily seen that the C^*-algebra $C^*(\Gamma)$ of a countable group Γ is residually finite dimensional if and only if every unitary representation of Γ in ℓ^2 can be approximated by representations of Γ which factor through representations of compact groups. Here the topology on the set of representations $\mathrm{Rep}\,(\Gamma, \ell^2)$ is that induced from the product topology on $U(\ell_2)^\Gamma$, where the unitary group $U(\ell^2)$ is equipped with the strong operator topology. Since the group $U(\ell^2)$ can be viewed as a group of isometries of the unit sphere $\mathbb{S}^\infty$ in ℓ^2, a way to a better understanding of the problem could be to enlarge the settting and consider representations of Γ by isometries of other metric spaces. Other than the sphere $\mathbb{S}^\infty$, a very natural candidate to consider is the

universal Urysohn metric space $\mathbb{U}$ [22, 23, 24], the object of considerable attention in recent years [12, 25, 29, 9, 7, 17]. Just like the sphere, the Urysohn space is an ultrahomogeneous complete separable metric space (that is, every isometry between two finite subspaces extends to a global self-isometry of the entire space), but unlike the sphere, $\mathbb{U}$ contains an isometric copy of every separable metric space.

For a topological group G and a discrete group Γ, denote by $\operatorname{Rep}(\Gamma, G)$ the space of all homomorphisms from Γ to G, considered as a subspace of G^{Γ} with the product topology. A representation of a group Γ by isometries of a metric space X is just a homomorphism $\Gamma \to \operatorname{Iso}(X)$, where we equip the latter group with the topology of pointwise convergence on X (it coincides with the compact-open topology). We will denote $\operatorname{Rep}(\Gamma, \operatorname{Iso}(X))$ simply by $\operatorname{Rep}(\Gamma, X)$.

Theorem 1. *Let Γ be a residually finite group. Every representation of Γ by isometries of the Urysohn space $\mathbb{U}$ is approximated by representations with a finite range. In other words, representations of Γ with a finite range are everywhere dense in the space* $\operatorname{Rep}(\Gamma, \mathbb{U})$.

Our method of proof further refines a construction from [16, 18]. Note again that for unitary representations an analogue of Theorem 1 does not hold, as follows from above mentioned Bekka's result from [1] together with Corollary 9 below.

Recall that the *lower Vietoris topology* on the set $\mathcal{F}(X)$ of all closed subsets of a topological space X is determined by the basic sets

$$L(V_1, V_2, \ldots, V_n) = \{F \in \mathcal{F}(X) \colon \forall i = 1, 2, \ldots, n, \ F \cap V_i \neq \emptyset\},$$

where V_i, $i = 1, 2, \ldots, n$ are open subsets of X.

Consider the following three properties of a topological group G:

($\circledast_1$) if A and B are finite subsets of G and every element of A commutes with every element of B, then there exist finite subsets A' and B' of G that are arbitrarily close to A and B, respectively, such that every element of A' commutes with every element of B' and the subgroups of G generated by A' and B' are relatively compact (equivalently, the subgroup generated by $A' \cup B'$ is relatively compact).

Here "arbitrarily close" means the following: if $A = \{x_1, \ldots, x_n\}$ and $Ox_1, \ldots, Ox_n$ are neighbourhoods of $x_1, \ldots, x_n$, then $A' = \{x'_1, \ldots, x'_n\}$, where $x'_i \in Ox_i$, and similarly for B'. If G is a metric group, then one can simply talk of closeness with regard to the Hausdorff metric. "Relatively compact" means "with a compact closure."

($\circledast_2$) For every pair H_1, H_2 of closed topological subgroups of G where every element of H_1 commutes with every element of H_2, there are nets $K_{1,\alpha}$, $K_{2,\alpha}$ of compact subgroups of G such that

- Every element of $K_{1,\alpha}$ commutes with every element of $K_{2,\alpha}$, and
- $K_{i,\alpha}$ converges to H_i in the lower Vietoris topology, $i = 1, 2$.

($\circledast_3$) Let F_∞ be the free group on countably many generators, $\Gamma = F_\infty \times F_\infty$. Then the homomorphisms $h \colon \Gamma \to G$ with a relatively compact range form a dense subset of the space $\operatorname{Rep}(\Gamma, G)$.

The reader will easily verify that the properties $\circledast_1$, $\circledast_2$, $\circledast_3$ are equivalent to each other. We confine ourselves by the following remark: every homomorphism of $\Gamma = F_\infty \times F_\infty$ to G gives rise to a pair of commuting subgroups of G, and every pair of commuting finitely-generated subgroups of G can be obtained in such a way.

Definition 2. A topological group G satisfies the *Kirchberg property* if it satisfies the equivalent properties $\circledast_1$, $\circledast_2$, $\circledast_3$ considered above.

By force of Kirchberg's work, Connes' Embedding Conjecture is equivalent to the Kirchberg property for the unitary group $G = U(\ell^2)$ with the strong topology. In our view, it makes sense to investigate the validity of the Kirchberg property for other infinite-dimensional topological groups of importance. In particular, it follows from Theorem 1 that the isometry group of the Urysohn space, which forms an infinite-dimensional group of considerable current interest, satisfies the property in question.

Corollary 3. *The isometry group* Iso $(\mathbb{U})$ *of the Urysohn metric space* $\mathbb{U}$*, equipped with the natural Polish topology (topology of simple convergence), satisfies the Kirchberg property.*

Our paper is organized as follows. The proof of our main Theorem 1, as well as that of Corollary 3, are contained in Section 3, while the necessary prerequisites about the Urysohn space and its group of isometries are collected in Section 2. Finally, in Section 4 we discuss in some detail Connes' Embedding Conjecture and the proposed Kirchberg property of topological groups.

2. Prerequisites on the Urysohn space

Recall that the universal Urysohn metric space $\mathbb{U}$ [25, 29, 9, 7, 17] is characterized by the following properties:
    (1) $\mathbb{U}$ is a complete separable metric space;
    (2) $\mathbb{U}$ contains an isometric copy of every separable metric space;
    (3) every isometry between two finite metric subspaces of $\mathbb{U}$ extends to a global isometry of $\mathbb{U}$ onto itself.
    Alternatively, $\mathbb{U}$ is the only complete separable metric space that is *finitely injective:* for every finite metric spaces $K \subset L$ every distance-preserving map $K \to \mathbb{U}$ has a distance-preserving extension $L \to \mathbb{U}$. As a topological space, $\mathbb{U}$ is the same as the Hilbert space ℓ^2 [28]. The group $P = \mathrm{Iso}\,(U)$ is a universal topological group with a countable base: every topological group with a countable base is isomorphic to a topological subgroup of P [25], [26], [27]. In this section we consider a few lemmas that will be used in Section 3.

Lemma 4. *Let G be a residually finite group, d a left-invariant pseudometric on G, K a finite subset of G. Suppose that the restriction of d to K is a metric. Then there exist a subgroup $H \subset G$ of finite index and a G-invariant pseudometric on the (finite) set G/H such that the restriction of the quotient map $G \to G/H$ to K is distance-preserving.*

This is essentially Lemma 2.5 of [18]. For the reader's convenience we give a proof.

Proof. A *seminorm* on a group is a non-negative function p satisfying the following: (1) $p(1) = 0$; (2) $p(xy) \leq p(x) + p(y)$; (3) $p(x^{-1}) = p(x)$. There is a one-to-one correspondence between left-invariant pseudometics and seminorms: given a seminorm p,

consider the pseudometric μ defined by $\mu(x, y) = p(x^{-1}y)$; given μ, we recover p by $p(x) = \mu(x, 1)$.

The values of pseudometrics and seminorms are (finite) non-negative real numbers, but in this proof it will be convenient to allow the value $+\infty$ as well. In that case we use the terms *pseudometric$_*$* and *seminorm$_*$*.

Let p be the seminorm corresponding to d, $p(x) = d(x, 1)$. Let $L = K^{-1}K \subset G$. Consider the greatest seminorm$_*$ q on G that agrees with p on L. This seminorm$_*$ is defined by

$$q(x) = \inf \sum_{i=1}^{n} p(y_i),$$

where the infimum is taken over all representations of the form $x = y_1 \ldots y_n$, $y_i \in L$, $1 \le i \le n$. (The infimum of the empty set is $+\infty$, thus q is infinite outside the subgroup generated by K.) As p is strictly positive on $L \setminus \{1\}$, for every $C > 0$ the set $G_C = \{x \in G : q(x) \le C\}$ is finite. Let $M = \max\{p(x) : x \in L\}$ and $C = 2M$. Since G is residually finite, there exists a normal subgroup $H \lhd G$ of finite index such that $H \cap G_C = \{1\}$. Consider the quotient seminorm$_*$ $\dot{q}$ on G/H, defined by

$$\dot{q}(xH) = \inf\{q(y) : y \in xH\}.$$

We claim that $\dot{q}(xH) = p(x)$ for every $x \in L$. Indeed, otherwise there exist $h \in H$ and a representation $xh = y_1 \ldots y_n$ with $y_1, \ldots, y_n \in L$ and $\sum p(y_i) < p(x)$. But then $h \ne 1$ and $q(h) = q(x^{-1}y_1 \ldots y_n) \le p(x) + \sum p(y_i) < 2p(x) \le C$, hence $h \in H \cap G_C$, in contradiction with our choice of H.

Equip G/H with the pseudometric$_*$ λ corresponding to $\dot{q}$. Then the quotient map $G \to G/H$ is distance preserving on K: if $x, y \in K$, then $x^{-1}y \in L$, so $\lambda(xH, yH) = \dot{q}(x^{-1}yH) = p(x^{-1}y) = d(x, y)$. If λ takes infinite values, replace it by $\inf(\lambda, M)$. $\square$

Lemma 5. *Let μ and λ be two pseudometrics on a finite set K. Let R be the diameter of the space (K, λ). For every distance-preserving map $i : (K, \mu) \to \mathbb{U}$ there exists a distance preserving map $j : (K, \mu + \lambda) \to \mathbb{U}$ such that $d(i(x), j(x)) \le R$ for every $x \in K$. Moreover, if $a \in K$ is given, we may require that $j(a) = i(a)$.*

Proof. Consider the product $K \times K$ with the pseudometric ν defined by

$$\nu((x, y), (x', y')) = \mu(x, x') + \lambda(y, y')$$

and the embeddings $i', j' : K \to K \times K$ defined by $i'(x) = (x, a)$ and $j'(x) = (x, x)$. Write i as $i = g \circ i'$, where $g : K \times \{a\} \to U$ is distance-preserving. Since U is finitely injective, there exists a distance-preserving map $h : K \times K \to U$ extending g. Set $j = h \circ j'$. Then $j(a) = h(a, a) = i(a)$ and $d(i(x), j(x)) = d(h(i'(x)), h(j'(x))) = \nu(i'(x), j'(x)) = \lambda(x, a) \le R$ for every $x \in K$. $\square$

Recall that the group $\text{Iso}(X)$ of isometries of a metric space is equipped with the topology of pointwise convergence on X, or, which is equivalent, the compact-open topology. If X is complete separable, the group $\text{Iso}(X)$ is Polish.

Following [26], let us say that a subspace $X \subset \mathbb{U}$ is *g-embedded* if there exists a homomorphism $k : \text{Iso}(X) \to \text{Iso}(\mathbb{U})$ of topological groups such that for every $h \in \text{Iso}(X)$ the isometry $k(h) \in \text{Iso}(\mathbb{U})$ is an extension of h.

Lemma 6. *Every finite subset of U is g-embedded.*

Proof. Every separable metric space admits a g-embedding in $\mathbb{U}$ [25, 26, 27]. Since any two embeddings of a finite metric space in $\mathbb{U}$ are conjugate by an isometry of U, the lemma follows. □

The same argument shows that "finite" can be replaced by "compact" in the lemma.

3. Proofs of the main results

Proof of Theorem 1. We denote the topological group $\mathrm{Iso}\,(\mathbb{U})$ by P. Let Γ be a residually finite group. Let $f : \Gamma \to P$ be a homomorphism, and let O be a neighborhood of f in $\mathrm{Rep}\,(\Gamma, P)$. We must find $f' \in O$ with a finite range.

We may assume that O has the following form: there are finite sets $K \subset \Gamma$ and $A \subset \mathbb{U}$ and $\varepsilon > 0$ such that any $h \in \mathrm{Rep}\,(\Gamma, P)$ is in O if and only if $h(g)x$ is ε-close to $f(g)x$ for every $g \in K$ and $x \in A$. (We say that two points are ε-*close* if the distance between them is $\leq \varepsilon$.) We may also assume that K contains the neutral element $1 \in \Gamma$.

Case 1. Suppose first that A is a singleton: $A = \{p\}$ for some $p \in \mathbb{U}$. Let d be the metric on $\mathbb{U}$. Consider the left-invariant pseudometric μ on Γ defined by $\mu(g_1, g_2) = d(f(g_1)p, f(g_2)p)$. Let λ be the discrete metric on Γ such that $\lambda(g, h) = 1$ for any pair of distinct elements $g, h \in \Gamma$. Let $\rho = \mu + \varepsilon\lambda$. This is a left-invariant metric on Γ. According to Lemma 4, there exists a subgroup $H \subset \Gamma$ of finite index such that the finite set $Y = \Gamma/H$ carries a Γ-invariant pseudometric extending the metric $\rho|K$, where K is identified with its image in Y. In virtue of lemma 5, the distance preserving map $g \mapsto f(g)p$ from (K, μ) to $\mathbb{U}$ can be approximated by a distance preserving map $j : (K, \rho) \to \mathbb{U}$ such that $j(1) = f(1)p = p$ and

$$d(j(g), f(g)p) \leq \mathrm{diam}(K, \varepsilon\lambda) = \varepsilon$$

for every $g \in K$. Using the injectivity of $\mathbb{U}$, extend j to a distance-preserving map (which we still denote by j) $Y \to \mathbb{U}$. The natural action by isometries of Γ on Y gives rise to an action by isometries of Γ on $j(Y)$ such that j becomes a morphism of Γ-spaces. In virtue of Lemma 6, there exists a homomorphism $k : \mathrm{Iso}\,(j(Y)) \to P = \mathrm{Iso}\,(\mathbb{U})$ which sends each isometry of $j(Y)$ to its extension over $\mathbb{U}$. Let $f' : \Gamma \to P$ be the composition of the homomorphism $\Gamma \to \mathrm{Iso}\,(j(Y))$ corresponding to the action of Γ on $j(Y)$, with k. Since the group $\mathrm{Iso}\,(j(Y))$ is finite, f' has a finite range. We claim that $f' \in O$. It suffices to verify that $f'(g)p$ and $f(g)p$ are ε-close with respect to d for every $g \in K$. The g-shift $j(x) \mapsto j(gx)$ on $j(Y)$ sends the point $p = j(1)$ to $j(g)$ which is ε-close to $f(g)p$. Since $f'(g)$ extends the g-shift on $j(Y)$, we have $f'(g)p = j(g)$, and thus $d(f'(g)p, f(g)p) = d(j(g), f(g)p) \leq \varepsilon$.

Case 2. If $A = \{p_1, \ldots, p_n\} \subset \mathbb{U}$ has more than one point, we replace Γ by the free product $\Gamma * F$, where F is the free group on n generators $b_1, \ldots, b_n$, and use a reduction to Case 1. Pick a point $p \in \mathbb{U}$, and let F act by isometries on $\mathbb{U}$ in such a way that the generator b_i sends p to p_i, $1 \leq i \leq n$. Use this action to extend in an obvious way the homomorphism $f : \Gamma \to P$ to a homomorphism (still denoted by f) $\Gamma * F \to P$. Free groups are residually finite, and the free product of residually finite groups is residually finite (cf. [10]), so we can apply the result of the preceding paragraph to the group $\Gamma * F$. Consider the finite subset $L = \{gb_i : g \in K, 1 \leq i \leq n\}$ of $\Gamma * F$. According to Case 1, there exists a homomorphism $f' : \Gamma * F \to P$ such that f' has a finite range and $f'(g)p$ is $\varepsilon/2$-close to $f(g)p$ for every $g \in L$. In particular,

$t_i = f'(b_i)p$ is $\varepsilon/2$-close to $f(b_i)p = p_i$. For every $g \in K$ and every $i = 1, \ldots, n$ the point $f'(g)p_i$ is $\varepsilon/2$-close to $f'(g)t_i$, while $f'(g)t_i = f'(g)f'(b_i)p = f'(gb_i)p$ is $\varepsilon/2$-close to $f(gb_i)p = f(g)f(b_i)p = f(g)p_i$. It follows that $f'(g)p_i$ and $f(g)p_i$ are ε-close. Thus the restriction of f' to Γ has the required properties. □

Proof of Corollary 3. The group $\Gamma = F_\infty \times F_\infty$ is residually finite. According to Theorem 1, representations with a finite range form a dense subspace of $\mathrm{Rep}\,(\Gamma, \mathbb{U})$. This is stronger than the Kirchberg property $\circledast_3$. □

4. Connes' Embedding conjecture and Kircherg's property of topological groups

In this section we discuss Connes' Embedding Conjecture (CEC), which is presently one of the main open problems in the theory of operator algebras. As mentioned in Section 1, CEC is equivalent to the conjecture that the unitary group $U_s(\ell^2)$ satisfies the Kirchberg property. Thus our Corollary 3 can be viewed as an analogue of CEC for the Urysohn space and its group of isometries.

Since this paper is aimed at experts in topological groups as much as at those in operator algebras, for the benefit of the former we begin by recalling some basic facts of the theory of operator algebras, referring the reader for a more detailed treatment, for instance, to the books by Sakai [20] and Takesaki [21].

Recall that a *von Neumann algebra* M is a unital C^*-algebra which, regarded as a Banach space, is a dual space: there is a (necessarily unique) Banach space M_*, the *predual* of M, with the property that M is isometrically isomorphic to $(M_*)^*$. A von Neumann algebra with a separable predual is called *hyperfinite* if it is generated, as a von Neumann algebra, by an increasing sequence of finite-dimensional subalgebras.

A von Neumann algebra M is called a *factor* if the centre of M is trivial, that is, consists of scalar multiples of 1. For example, the von Neumann algebra $\mathcal{L}(\ell^2)$ of all bounded linear operators on the Hilbert space ℓ^2 is a hyperfinite factor.

Let $E_\alpha, \alpha \in A$ be a family of normed spaces, and let ξ be an ultrafilter on the index set A. The (*Banach space*) *ultraproduct* of the family (E_α) along the ultrafilter ξ is the linear space quotient of the ℓ^∞-type direct sum $E = \oplus_{\alpha \in A}^{\ell^\infty} E_\alpha$ by the ideal $\mathcal{I}_\xi$ formed by all collections $(x_\alpha)_{\alpha \in A} \in E$ with the property

$$\lim_{\alpha \to \xi} x_\alpha = 0,$$

equipped with the norm

$$\|x\| = \lim_{\alpha \to \xi} x_\alpha,$$

where (x_α) is any representative of the equivalence class x. If the ultrafilter ξ is free, the ultraproduct is always a Banach space. For a general theory of ultraproducts of normed spaces (also known in nonstandard analysis as *nonstandard hulls*), see [11].

The ultrapower of a family of C^*-algebras is again a C^*-algebra, but the property of being a factor is not necessarily preserved. However, in the particular case where all factors in a family are the so-called finite factors, one can modify the construction of an ultraproduct so as to obtain a factor.

Recall that a (finite) *trace* on a von Neumann algebra M is a positive linear functional $\tau \colon M \to \mathbb{C}$ with the property $\tau(AB) = \tau(BA)$ for all $A, B \in M$. A trace τ is *normalized* if $\tau(1) = 1$. One says that a factor M is *finite* if it admits a trace. One

can show that in this case the normalized trace on M is unique. Finite factors of finite dimension are exactly all matrix algebras of the form $M_n(\mathbb{C})$, $n \in \mathbb{N}$. However, there exist finite factors that are infinite-dimensional as normed spaces. They are called *factors of type* II_1.

An example is given by the following construction. Let G be a (countable) discrete group. Denote by $VN(G)$ the strongly closed unital $*$-subalgebra of $\mathcal{L}(\ell^2(G))$ generated by all operators of left translation by elements of G. This is the so-called (*reduced*) *group von Neumann algebra* of G. If all conjugacy classes of G except for that of unity are infinite, then $VN(G)$ is a factor of type II_1. For example, this is the case where $G = F_2$, the free group on two generators. On the contrary, the factor $\mathcal{L}(\ell^2)$ does not admit a trace.

As was shown by Murray and von Neumann, there exists only one, up to an isomorphism, hyperfinite factor of type II_1, denoted by R. For instance, R is isomorphic to the group von Neumann algebra of a locally finite group (the union of an increasing sequence of finite subgroups) with infinite conjugacy classes. (Probably the simplest example is S_∞^{fin}, the group of all permutations of the set $\mathbb{N}$ of natural numbers having a finite support each.)

Now let M_α be a family of finite factors, each equipped with a normalized trace τ_α, and let ξ be an ultrafilter on the index set A. The formula

$$\tau((x_\alpha)) = \lim_{\alpha \to \xi} \tau_\alpha(x_\alpha)$$

determines a trace on the Banach space ultraproduct M of the family (M_α) along ξ. The subset

$$\mathcal{I}_\xi = \{x \in M : \tau(x^*x) = 0\}$$

is an ideal of M, and the factor-algebra $M/\mathcal{I}_\xi$ happens to be a finite factor, called the *von Neumann ultraproduct* of the family (M_α). Under an obvious non-degeneracy assumption (for every $n \in \mathbb{N}$, the set $\{\alpha \in A: \dim(M_\alpha) \geq n\}$ is in ξ), the von Neumann ultraproduct $M/\mathcal{I}_\xi$ is non-separable, thus has infinite dimension and is a factor of type II_1. For instance, the von Neumann ultraproduct of all matrix algebras $M_n(\mathbb{C})$, $n \in \mathbb{N}$, equipped with their standard normalized traces, along any free ultrafilter on the natural numbers, is a factor of type II_1.

As every subfactor of a factor of type II_1 is again of type II_1, one may wonder how large is the class of all separable factors of type II_1 embeddable into ultrapowers of R. Such factors do not need to be hyperfinite: already Connes had remarked [3] that $VN(F_2)$ is among them.

The following conjecture was formulated by Connes in the same paper [3] (p. 105, third paragraph from the bottom).

Connes' Embedding Conjecture. *Every factor of type* II_1 *embeds into an ultrapower of the hyperfinite factor* R *of type* II_1.

In the above conjecture, one can assume without loss in generality that the factors have separable preduals, and the index set supporting the ultrafilter is countable. Furthermore, one can replace the ultrapower of R with the von Neumann ultraproduct of matrix algebras $M_n(\mathbb{C})$. For a discussion, see e.g. section 9.10 in [19].

In the last three decades, the conjecture has increased in importance and has become one of the main open problems of operator algebras theory. Largely through

the work of E. Kirchberg, numerous equivalent forms of Connes' conjecture came into existence.

By a C^*-*norm* on $*$-algebra we mean a norm *the completion* with respect to which is a C^*-algebra. If A and B are two unital C^*-algebras, their algebraic tensor product $A \otimes B$ is not, in general, a C^*-algebra again (unless one of the algebras is finite-dimensional), but it always admits at least one C^*-norm extending the norms on A and B (we identify A and B with their images under the natural embeddings $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$). For instance, if A is a C^*-subalgebra of $\mathcal{L}(\mathcal{H}_1)$ and B is a a C^*-subalgebra of $\mathcal{L}(\mathcal{H}_2)$, then $A \otimes B$ embeds naturally as a C^*-subalgebra into $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (the tensor product of Hilbert spaces), and the norm induced by this embedding is called the *minimal tensor product norm.* It has the remarkable property of being smaller than any other C^*-norm on $A \otimes B$ (Takesaki). On the other hand, there exists the *maximal tensor product norm,* which is the largest among all C^*-norms on $A \otimes B$. The minimal and maximal tensor product norms on $A \otimes B$ coincide in a number of important cases, for instance, when one of the algebras A, B is *nuclear.*

If no confusion can arise, one normally denotes the C^*-algebra completion of the algebraic tensor product $A \otimes B$, equipped with one or another norm, with the same symbol $\otimes$, without adding a "hat" to mark the completion, for instance, $A \otimes_{min} B$ is the *completion* of the algebraic tensor product equipped with the minimal tensor norm, and so forth.

If A is a unital C^*-algebra, the *unitary group* of A is a multiplicative subgroup consisting of all unitaries of A, that is, $u \in A$ with $u^*u = uu^* = 1$. Every discrete group G admits a universal embedding, as a subgroup, into the unitary group of a suitable C^*-algebra. Namely, there exist a unital C^*-algebra $C^*(G)$, called the (*full*) *group C^*-algebra* of G, and a group homomorphism (in fact, a monomorphism), i, from G to the unitary group $U(C^*(G))$ with the property that, whenever A is a unital C^*-algebra and $f \colon G \to U(A)$ is a group homomorphism, there is a unique morphism of C^*-algebras $\bar{f} \colon C^*(G) \to A$ with $\bar{f} \circ i = f$. The C^*-algebra $C^*(G)$ is the C^*-envelope of the $*$-algebra $l^1(G)$, it is unique up to an isomorphism for every discrete group G.

Here is a useful example to consider: the full C^*-algebra of the direct product $G \times H$ of two groups is naturally isomorphic to the *maximal* tensor product $C^*(G) \otimes_{max} C^*(H)$.

Below is a statement which is equivalent to the Connes' Embedding Conjecture [13], see also [15] or [19], ch. 16.

Conjecture (Kirchberg) *The tensor product of the group C^*-algebra $C^*(F_2)$ of the free group on two generators with itself admits a unique C^*-algebra norm. (That is, the max and min norms on $C^*(F_2) \otimes C^*(F_2)$ coincide.)*

A *representation* of a C^*-algebra A in a Hilbert space $\mathcal{H}$ is a C^*-algebra morphism $\pi \colon A \to \mathcal{L}(\mathcal{H})$. The *essential space* of a representation π is the closure of $\pi(A)(\mathcal{H})$ in $\mathcal{H}$. A representation π is *degenerate* if its essential space is a proper subspace of $\mathcal{H}$, and *finite-dimensional* if the essential space is finite-dimensional. A representation π of a unital C^*-algebra is *unital* if $\pi(1) = \mathbb{I}_{\mathcal{H}}$. If A is a unital C^*-algebra, then a representation π of A is unital if and only if it is non-degenerate.

A C^*-algebra A is called *residually finite-dimensional* (*RFD*) if it admits a separating family of finite-dimensional representations. For instance, the full group C^*-algebra $C^*(F)$ of the non-abelian free group (on any number of generators) is RFD, this is a result by Choi [2]. Strictly speaking, the finite-dimensionality of the algebra $\pi(A)$ is necessary, but not sufficient, for π to be finite-dimensional: the representation of the one-dimensional C^*-algebra $\mathbb{C}$ in ℓ^2 given by $\pi(\lambda) = \lambda\mathbb{I}$ has all of ℓ^2 as its essential space.

At the same time, a (unital) algebra A is RFD if and only if it admits a separating family of (unital) representations with finite-dimensional image, simply because every finite-dimensional algebra admits a faithful finite-dimensional representation.

It is not difficult to verify that the *minimal* tensor product of two residually finite-dimensional C^*-algebras is again residually finite-dimensional, and also that if the *maximal* tensor product of two C^*-algebras is residually finite-dimensional, then the maximal norm on the tensor product coincides with the minimal one. These observations lead to the following further reformulation of CEC, noted for example by Ozawa [15], Prop. 3.19.

Conjecture (equivalent to CEC). *The group C^*-algebra $C^*(F_2 \times F_2)$ is residually finite dimensional.*

While the group $F_2 \times F_2$ is of course residually finite, residual finiteness of a group Γ is in general insufficient for the algebra $C^*(\Gamma)$ to be RFD [1].

If A is a C^*-algebra, then $\operatorname{Rep}(A, \mathcal{H})$ stands for the set of all (degenerate and non-degenerate) representations of A in $\mathcal{H}$. Following Exel and Loring [5], equip the set $\operatorname{Rep}(A, \mathcal{H})$ with the coarsest topology making all the mappings of the form

(1) $$\operatorname{Rep}(A, \mathcal{H}) \ni \pi \mapsto \pi(x)(\xi) \in \mathcal{H}, \ x \in A, \ \xi \in \mathcal{H}$$

continuous. Clearly, this topology is inherited from $C_p(A, \mathcal{B}_s(\mathcal{H}))$; here the subscript "$p$" as usual, stands for the topology of pointwise convergence, while $\mathcal{B}_s(\mathcal{H})$ is the space $\mathcal{B}(\mathcal{H})$ endowed with the strong operator topology, that is, the topology induced from $C_p(\mathcal{H}, \mathcal{H})$. The basic neighbourhoods of an element $\pi \in \operatorname{Rep}(A, \mathcal{H})$ are of the form

$$\mathcal{O}_\pi[x_1, x_2, \ \ldots \ , x_n; \Xi; \varepsilon]$$
$$= \ \{\eta \in \operatorname{Rep}(A, \mathcal{H}) \colon \|\pi(x_i)(\xi) - \eta(x_i)(\xi)\| < \varepsilon, \ i = 1, 2, \ldots, n, \ \xi \in \Xi\},$$

where $x_i \in A$ and Ξ is a finite system of vectors in $\mathcal{H}$.

Theorem 7 (Exel and Loring [5]). *A C^*-algebra A is residually finite-dimensional if and only if the set of finite-dimensional representations is everywhere dense in $\operatorname{Rep}(A, \mathcal{H})$ for all Hilbert spaces $\mathcal{H}$.*

Every group C^*-algebra, $C^*(\Gamma)$, admits a *counit*, that is, a one-dimensional unital representation η, in $\mathcal{H}$, which is determined by the condition $\eta(g) = \mathbb{I}$ for all $g \in \Gamma$. Let $\pi \in \operatorname{Rep}(C^*(\Gamma), \mathcal{H})$. Denote $\mathcal{H}_1 = \pi(C^*(\Gamma))(\mathcal{H})$ and associate to π the representation $\tilde{\pi} = (\pi|\mathcal{H}_1) \oplus \eta_{\mathcal{H} \ominus \mathcal{H}_1}$. This is a unital representation of $C^*(\Gamma)$ in $\mathcal{H}$, and if π is finite-dimensional, then $\tilde{\pi}$ has a finite-dimensional image. Notice that π itself can be written in the form $\tilde{\pi} = (\pi|\mathcal{H}_1) \oplus \mathbf{O}_{\mathcal{H} \ominus \mathcal{H}_1}$.

For a unital C^*-algebra A, denote by $\operatorname{Rep}_1(A, \mathcal{H})$ the subspace of $\operatorname{Rep}(A, \mathcal{H})$ consisting of unital representations.

Corollary 8. *A group C^*-algebra $A = C^*(\Gamma)$ is residually finite-dimensional if and only if the set of unital representations with finite-dimensional image is everywhere dense in* $\mathrm{Rep}_1(A, \mathcal{H})$ *for all Hilbert spaces $\mathcal{H}$.*

Proof. $\Rightarrow$: if a representation $\pi \in \mathrm{Rep}_1(A, \mathcal{H})$ is approximated by a net of finite-dimensional representations (π_α), then π is clearly approximated by the net $(\widetilde{\pi_\alpha})$ of unital representations with finite-dimensional images.

$\Leftarrow$: let $\pi \in \mathrm{Rep}(A, \mathcal{H})$ be arbitrary. We want to approximate π with finite-dimensional representations. Without loss in generality, we may assume that π is non-degenerate and so unital. There is a net (π_α) of unital representations with finite-dimensional images approximating $\tilde{\pi}$. Let $x_1, x_2, \ldots, x_n \in A$, let $\Xi \in \mathcal{H}$ be finite, and let $\varepsilon > 0$. Find an α with

$$\|\pi(x_i)(\xi) - \pi_\alpha(x_i)(\xi)\| < \varepsilon, \ i = 1, 2, \ldots, n, \ \xi \in \Xi.$$

Denote by $\mathcal{H}_1$ the linear subspace of $\mathcal{H}$ spanned by elements $\pi_\alpha(x)(\xi)$, $x \in A$, $\xi \in \Xi$. This $\mathcal{H}_1$ is finite-dimensional and invariant under all operators in $\pi_\alpha(A)$. The restriction $\dot{\pi}_\alpha$ of π_α to $\mathcal{H}_1$ is a finite-dimensional representation of A, and

$$\|\pi(x_i)(\xi) - \dot{\pi}_\alpha(x_i)(\xi)\| < \varepsilon, \ i = 1, 2, \ldots, n, \ \xi \in \Xi.$$

$\square$

Unital representations of the group C^*-algebra $C^*(G)$ in a Hilbert space $\mathcal{H}$ are in a natural one-to-one correspondence with the unitary representations of the group G in $\mathcal{H}$, that is, group homomorphisms from G to the unitary group $U(\mathcal{H})$ of the Hilbert space $\mathcal{H}$. We will equip the latter group with the *strong operator topology*. This is simply the topology of pointwise convergence on $\mathcal{H}$ (or on the unit ball), that is, a topology induced by the embedding $U(\mathcal{H}) \subseteq C_p(\mathcal{H}, \mathcal{H})$. This topology makes $U(\mathcal{H})$ into a Polish topological group.

In view of the above remarks, there is a canonical bijection $\mathrm{Rep}_1(C^*(G), \mathcal{H}) \leftrightarrow \mathrm{Rep}(G, \mathcal{H})$, which is in fact a homeomorphism.

The image $\pi(G)$ of a representation π of a group G is a topological subgroup of the unitary group $U(\mathcal{H})$ with the strong operator topology. This image is a relatively compact subgroup if and only if π factors through a strongly continuous representation of a compact group. For instance, this is the case where π is the restriction to G of a representation of $C^*(G)$ with a finite-dimensional image. Since every strongly continuous representation of a compact group, on the other hand, decomposes into a direct sum of finite-dimensional representations, one can easily deduce the following.

Corollary 9. *For a discrete group G, the following conditions are equivalent:*
1. *the full C^*-algebra $C^*(G)$ is residually finite-dimensional;*
2. *if $\mathcal{H}$ is a Hilbert space, representations with relatively compact image (in the strong topology) are everywhere dense in the space $\mathrm{Rep}(G, \mathcal{H})$.*

Here it is enough to take a Hilbert space $\mathcal{H}$ of the same density character as the cardinality of G.

The topology considered by Exel and Loring is finer than the well-known *Fell topology* [6]. For the Fell topology, an analogue of the above characterization can be found in [4].

We now summarize our discussion of various equivalent forms of CEC:

Theorem 10. *Each of the following conjectures is equivalent to Connes' Embedding Conjecture:*

 (1) *the algebra $C^*(F_\infty \times F_\infty)$ is residually finite-dimensional;*
 (2) *the algebra $C^*(F_2 \times F_2)$ is residually finite-dimensional;*
 (3) *the unitary group $U_s(\ell^2)$ has the Kirchberg property.*

The equivalence of (1) and (2) between themselves follows from the fact that the algebras $C^*(F_\infty \times F_\infty)$ and $C^*(F_2 \times F_2)$ are unital C^*-subalgebras of each other. The equivalence of (1) and (2) to Connes' Embedding Conjecture is Kirchberg's difficult result [13], cf. also [15, 19], while the equivalence of (1) and (3) follows from Corollary 9, applied to the group $G = F_\infty \times F_\infty$.

Applying the property $\circledast_2$ with $H_2 = \{e\}$, one gets the following necessary condition for Kirchberg's property of a topological group G: every finite subset of G can be simultaneously approximated by elements of a finite set contained in a compact subgroup. In particular, this is the case if G admits an increasing chain of compact subgroups with everywhere dense union. This property is observed quite often in concrete "large" topological groups of importance. Clearly, the unitary group $U(\ell^2)$ is one of them. As shown by A.S. Kechris (private communication), an even stronger result holds: every finite collection of elements of $U(\ell^2)$ is simultaneously approximated with elements of a finite subgroup. The group $\mathrm{Iso}\,(U)$ admits an increasing chain of finite subgroups with an everywhere dense union [30], cf. also [18].

Other Polish groups approximable by increasing chains of compact subgroups include the infinite symmetric group S_∞ of all self-bijections of a countably infinite set and the group $\mathrm{Aut}\,(\mathbb{I}, \lambda)$ of measure-preserving transformations of the standard Borel space with a non-atomic probability measure. For more examples see e.g. [8, 17].

Theorem 1 allows one to conclude that the Polish group $\mathrm{Iso}\,(\mathbb{U})$ has the Kirchberg property (Corollary 3) and leads us to ask the following: does every Polish group G approximated by an increasing chain of compact subgroups satisfy Kirchberg's property?

If true, this will imply Connes' Embedding Conjecture when applied to $G = U(\ell^2)$. At present, we are unaware of any topological group counterexamples.

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